SURFACE AREA OF ELLIPSOIDS

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A . We study the surface area of an ellipsoid in $\mathbb{E}^n$ as the function of the lengths of their major semi-axes. We write down an explicit formula as an integral over $\mathbb{S}^{n-1}$, use this formula to derive convexity properties of the surface area, to greatly sharpen the estimates given in [5] for the surface area of a large-dimensional ellipsoid, to produce asymptotic formulas for the surface area and the *isoperimetric ratio* of an ellipsoid in large dimensions, and to give an expression for the surface in terms of the Lauricella hypergeometric function.

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In [5] estimates were given for the *mean curvature integrals* of an ellipsoid E in $\mathbb{E}^n$ in terms of the lengths of its *major semiaxes* – the 0-th mean curvature integral is simply the surface area of the ellipsoid E. The estimate for the surface area was simply the $n-1$-th symmetric function of the semi-axes, and it was shown in [5] that this estimate differed from the truth by a factor bounded only by a function of n (unfortunately, this function was of the order of magnitude $n^{n/2}$).

In this paper we write down a formula ((3)) expressing the surface area of E in terms of an integral of a simple function over the sphere $\mathbb{S}^{n-1}$. This formula will be used to deduce a number of results:

(1) The ratio of the surface area to the volume of E (call this ratio $\mathcal{R}(E)$) is a *norm* on the vectors of inverse semi-axes. (Theorem 1).
(2) By a simple transformation (introduced for this purpose in [4], though doubtlessly known for quite some time) $\mathcal{R}(E)$ can

Date: February 1, 2008.
1991 *Mathematics Subject Classification.* 52A38; 60F99; 58C35.
Key words and phrases. ellipsoid, surface area, expectation, integration, large dimension, Lindberg conditions, Lauricella hypergeometric function, harmonic mean.
The author is supported by the NSF DMS. Parts of this paper appeared as the preprint [4]; the author would like to thank Warren D. Smith on comments on a previous version of this paper.The author would also like to thank Princeton University for its hospitality.

be expressed as a moment of a sum of independent Gaussian random variables; this transformation can be used to evaluate or estimate quite a number of related spherical integrals (see Section 2).

(3) Quite sharp bounds ((15)) on the ratio of $\mathcal{R}(E)$ to the L^2 norm of the vectors of inverses of semi-axes are derived.

(4) We write down a very simple asymptotic formula (Theorem 11) for the surface area of an ellipsoid of a very large dimension with "not too different" axes. In particular, the formula holds if the ratio of the lengths of any two semiaxes is bounded by some fixed constant (Corollary 12).

(5) Finally, we give an identity relating the surface area of E to a linear combination of Lauricella hypergeometric functions.

Notation. Let (S, μ) be a measure space with $\mu(S) < \infty$. We will use the notation

$$\fint_S f(x)\, d\mu \stackrel{\text{def}}{=} \frac{1}{\mu(S)} \int_S f(x) d\mu.$$

In addition, we shall denote the area of the unit sphere $\mathbb{S}^n$ by ω_n and we shall denote the volume of the unit ball $\mathbb{B}^n$ by κ_n.

1. C ’

Let K be a convex body in $\mathbb{E}^n$. Let $u \in \mathbb{S}^{n-1}$ be a unit vector, and let us define $V_u(K)$ to be the (unsigned) $n-1$-dimensional volume of the orthogonal projection of K in the direction U. *Cauchy's formula* (see [6, Chapter 13]) then states that

$$\boxed{V_{n-1}(\partial K) = \frac{n-1}{\omega_{n-2}} \int_{\mathbb{S}^{n-1}} V_u(K)\, d\sigma = (n-1)\frac{\omega_{n-1}}{\omega_{n-2}} \fint_{\mathbb{S}^{n-1}} V_u(K)\, d\sigma,} \tag{1}$$

where $d\sigma$ denotes the standard area element on the unit sphere.

In the case where $K = E$ is an ellipsoid, given by

$$E = \{\mathbf{x} \in \mathbb{E}^n \quad | \quad \sum_{i=1}^n q_i^2 x_i^2 =\leq 1\}$$

there are several ways of computing V_u, one such way can be found in [1]. In any event, the result is:

$$V_u(E) = \kappa_{n-1} \frac{\sqrt{\left(\sum_{i=1}^n u_i^2 q_i^2\right)}}{\prod_{i=1}^n q_i}. \tag{2}$$

Since

$$V_n(E) = \frac{\kappa_n}{\prod_{i=1}^n q_i},$$

we can rewrite Cauchy's formula (1) for E in the form:

$$\boxed{\mathbb{R}(E) \stackrel{\text{def}}{=} \frac{V_{n-1}(\partial E)}{V_n(E)} = n \fint_{\mathbb{S}^{n-1}} \sqrt{\sum_{i=1}^n u_i^2 q_i^2}\, d\sigma,} \tag{3}$$

where $\mathbb{R}(E)$ is the *isoperimetric ratio* of E.

Theorem 1. *The ratio* $\mathbb{R}(E)$ *is a* norm *on the vectors* q *of lengths of semiaxes* ($q = (q_1, \dots, q_n)$.)

Proof. The integrand in the formula (3) is a norm. □

Corollary 2. *There exist constants* $c_{n,p}, C_{n,p}$*, such that*

$$c_{n,p}\|q\|_p \leq \mathbb{R}(q) \leq C_{n,p}\|q\|_p,$$

where $\|q\|_p$ *is the* L^p *norm of* q.

Proof. Immediate (since all norms on a finite-dimensional Banach space are equivalent). □

In the sequel we will find reasonably good bounds on the constants $c_{n,p}$ and $C_{n,p}$, but for the moment observe that if $a_i = 1/q_i, \quad i = 1, \dots, n$, then

$$\|q\|_p = \prod_{i=1}^n q_i \sigma_{n-1}^{1/p}(a_1^p, \dots, a_n^p), \tag{4}$$

where σ_{n-1} is the $n-1$-st elementary symmetric function. In particular, for $p = 1$, Corollary 2 together with Eq. (4) gives the estimate of [5] (only for the 0-th mean curvature integral and with (for now) ineffective constants – the latter part will be remedied directly). To exploit the formula (3) fully, we will need a digression on computing spherical integrals.

2. S

In this section we will prove the following easy but very useful Theorem:

Theorem 3. *Let* $f(x_1, \dots, x_n)$ *be a homogeneous function on* $\mathbb{E}^n$ *of degree* d *(in other words,* $f(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n)$*.) Then*

$$\Gamma\left(\frac{n+d}{2}\right) \fint_{\mathbb{S}^{n-1}} f d\sigma = \Gamma\left(\frac{n}{2}\right) \mathbb{E}\left(f(\mathbf{X}_1, \dots, \mathbf{X}_n)\right),$$

where $\mathbf{X}_1, \dots, \mathbf{X}_n$ *are independent random variables with probability density* e^{-x^2}.

Proof. Let

$$E(f) = \fint_{\mathbb{S}^{n-1}} f(x)\, d\sigma,$$

and let $N(f)$ be defined as $\mathbb{E}(f(\mathbb{X}_1, \dots, \mathbb{X}_\ltimes))$, where $\mathbb{X}_i$ is a Gaussian random variable with mean 0 and variance 1/2, (so with probability density $\mathfrak{n}(x) = e^{-x^2}$,) and $\mathbb{X}_1, \dots, \mathbb{X}_n$ are independent. By definition,

$$N(f)(n) = c_n \int_{\mathbb{E}^n} \exp\left(-\sum_{i=1}^n x_i^2\right) f(x_1, \dots, x_n)\, dx_1 \dots dx_n, \tag{5}$$

where c_n is such that

$$c_n \int_{\mathbb{E}^n} \exp\left(-\sum_{i=1}^n x_i^2\right) dx_1 \dots dx_n = 1. \tag{6}$$

We can rewrite the expression (5) for $N(f)$ in polar coordinates as follows (using the homogeneity of f):

(7)

$$\text{Nmin}\,(n) = c_n \mathrm{vol}\, \mathbb{S}^{n-1} \int_0^\infty e^{-r^2} r^{n+d-1} E(f)\, dr = c_n E(f) \int_0^\infty e^{-r^2} r^{n+d-1}\, dr.$$

Since, by the substitution $u = r^2$,

$$\int_0^\infty e^{-r^2} r^{n+d-1}\, dr = \frac{1}{2} \int_0^\infty e^{-u} u^{(n+d-2)/2} du = \frac{1}{2} \Gamma\left(\frac{n+d}{2}\right).$$

and Eq. (6) can be rewritten in polar coordinates as

$$1 = c_n \mathrm{vol}\, \mathbb{S}^{n-1} \int_0^\infty r^{n-1} dr = \frac{c_n \mathrm{vol}\, \mathbb{S}^{n-1}}{2} \Gamma\left(\frac{n}{2}\right),$$

we see that

$$\Gamma\left(\frac{n+d}{2}\right) E(f) = \Gamma\left(\frac{n}{2}\right) N(f).$$

□

Remark 4. In the sequel we will frequently be concerned with asymptotic results, so it is useful to state the following asymptotic formula (which follows immediately from Stirling's formula):

$$\lim_{x \to \infty} \frac{\Gamma(x+y)}{\Gamma(x)(x+y)^y} = 1. \tag{8}$$

It follows that for large n and fixed d,

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)} \sim \left(\frac{2}{n+d}\right)^{d/2}. \tag{9}$$

3. A

The Theorem in the preceding section can be used to give explicit formulas for the surface area of an ellipsoid (this formula will not be used in the sequel, however). Specifically, in the book [3] there are formulas for the moments of of random variables which are quadratic forms in Gaussian random variables. We know that for our ellipsoid E,

$$\mathbb{R}(E) = n \fint_{\mathbb{S}^{n-1}} \sqrt{\sum_{i=1}^n u_i^2 q_i^2}\, d\sigma = n \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \mathbb{E}\left(\sqrt{q_1^2 \mathbb{X}_1 + \cdots + q_n^2 \mathbb{X}_n}\right),$$

where $\mathbb{X}_i$ is a Gaussian with variance $1/2$. The expectation in the last expression is the $1/2$-th moment of the quadratic form in Gaussian random variables, and so the results of [3, p. 62] apply verbatim, so that we obtain:

$$\mathbb{R}(E) = n \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sqrt{\alpha} \int_0^\infty \frac{1}{\sqrt{z}} \sum_{j=1}^n \frac{q_j^2}{2(1+\alpha z q_j^2)} \left(\prod_{j=1}^n (1 - q_j^2 z)\right)^{-1/2} dz; \tag{10}$$

note that α in the above formula can be any positive number (as long as $|1 - \alpha q_j^2| < 1$, for all j.

This can also be expressed in terms of special functions. First, we need a definition:

Definition 5. *Let $a, b_1, \ldots, b_n, c, x_1, \ldots, x_n$ be complex numbers, with $|x_i| < 1, \quad i = 1, \ldots, n$, $\Re a > 0$, $\Re(c - a) > 0$. We then define the* Lauricella Hypergeometric Function $F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$ *as follows:*

$$F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} \prod_{i=1}^n (1 - u x_i)^{-b_i}\, du. \tag{11}$$

We also have the series expansion:

$$F_D(a;b_1,\dots,b_n;c;x_1,\dots,x_n) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}\prod_{i=1}^n(b_i)_{m_i}}{(c)_{m_1+\cdots+m_n}}\prod_{i=1}^n\frac{x_i^{m_i}}{m_i!}, \tag{12}$$

valid whenever $|x_i| < 1, \forall i$.

Now, we can write

$$\mathbb{R}(E) = n\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n+1}{2}\right)}\sqrt{\alpha}\times \sum_{j=1}^n \frac{q_j^2}{2}F_D\left(1/2;\eta_{1j},\dots,\eta_{nj};\frac{n+1}{2};1-\alpha q_1^2,\dots,1-\alpha q_n^2\right), \tag{13}$$

where $\eta_{ij} = 1/2+\delta_{ij}$, and α is a positive parameter satisfying $|1-\alpha q_j^2| < 1$.

4. L

Many of the results in this section will require the following basic lemmas.

Lemma 6. *Let $F_1,\dots,F_n,\dots$ be a sequence of probability distributions whose first moments converge to μ and whose second moments converge to* 0. *then F_i converge to the Dirac delta function distribution centered on μ.*

Proof. Follows immediately from Chebyshev's inequality. □

Lemma 7. *Suppose the distributions $F_1,\dots,F_n,\dots$ converge to the distribution F, and the expectations of $|x|^\alpha$ with respect to $F_1,\dots,F_n,\dots$ are bounded. Then the expectation of $|x|^\beta$, $0 \le \beta < \alpha$ converges to the expectation of $|x|^\beta$ with respect to F.*

Proof. See [2, pp. 251-252]. □

Theorem 8. *Let $\mathbb{Y}_1,\dots,\mathbb{Y}_n,\dots$ be* independent *random variables with means $0 < \mu_1,\dots,\mu_n,\dots < \infty$ and variances $\sigma_1^2,\dots,\sigma_n^2,\dots < \infty$ such that*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n\sigma_i^2}{\left(\sum_{i=1}^n\mu_i\right)^2} = 0. \tag{14}$$

Then

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{\mathbb{Y}_1+\cdots+\mathbb{Y}_n}{\sum_{i=1}^n\mu_i}\right)^\alpha = 1,$$

for $\alpha < 2$.

Proof. Consider the variable

$$\mathbb{Z}_n = \frac{\sum_{i=1}^n \mathbb{Y}_i}{\sum_{i=1}^n \mu_i}.$$

It is not hard to compute that

$$\sigma^2(\mathbb{Z}_n) = \frac{\sum_{i=1}^n \sigma_i^2}{\left(\sum_{i=1}^n \mu_i\right)^2},$$

while

$$\mu(\mathbb{Z}_n) = 1,$$

so by assumption (14) and Lemma 6 $\mathbb{Z}_n$ converges in distribution to the delta function centered at 1. The conclusion of the Theorem then follows from Lemma 7. □

Lemma 9. *Let* $\mathbb{X}$ *be normal with mean* 0 *and variance* 1/2 *(so probability density* $e^{-x^2}/\sqrt{\pi}$*.) Then*

$$\mathbb{E}(|\mathbb{X}|^p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}.$$

Proof.

$$\mathbb{E}(|\mathbb{X}|^p) = \frac{2}{\sqrt{\pi}} \int_0^\infty x^p e^{-x^2}\, dx = \frac{1}{\sqrt{\pi}} \int_0^\infty u^{(p-1)/2} e^{-u} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}.$$

□

Theorem 10.

$$⨍_{\mathbb{S}^{n-1}} \|u\|_p \, d\sigma \sim \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \left(n \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \right)^{\frac{1}{p}}.$$

Proof. This follows immediately from the 1-homogeneity of the L^p norm, the results of Section 2, Theorem 8, and Lemma 9. □

4.1. **Asymptotics of** $\mathbb{R}(E)$**..**

Theorem 11. *Let* $q_1, \ldots, q_n, \ldots$ *be a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n q_i^4}{\left(\sum_{i=1}^n q_i^2\right)^2} = 0.$$

Let E_n be the ellipsoid in $\mathbb{E}^n$ with major semiaxes $a_1 = 1/q_1, \dots, a_n = 1/q_n$. Then

$$\lim_{n\to\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\mathbb{R}(E_n)}{n\sqrt{\frac{1}{2}\sum_{i=1}^n q_i^2}} = 1.$$

Proof. The Theorem follows immediately from Theorem 8 and the results of Section 2. □

Corollary 12. *Let $a_1, \dots, a_n, \dots$ be such that $0 < c_1 \leq a_i/a_j \leq c_2 < \infty$, for any i, j. Let E_n be the ellipsoid with major semi-axes $a_1, \dots, a_n$. Then*

$$\lim_{n\to\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\mathbb{R}(E_n)}{n\sqrt{\frac{1}{2}\sum_{i=1}^n \frac{1}{a_i^2}}} = 1.$$

Proof. The quantities $q_1 = 1/a_1, \dots, q_n = 1/a_n, \dots$ clearly satisfy the hypotheses of Theorem 11 □

5. G $\mathbb{R}(E)$

We know that $\mathbb{R}(E)$ is a norm on the vector $\mathbf{q} = (q_1, \dots, q_n)$ – let us agree to write

$$||\mathbf{q}||_{\mathbb{R}} \stackrel{\text{def}}{=} \frac{\mathbb{R}(E)}{n} = \fint_{\mathbb{S}^{n-1}} \sqrt{\sum_{i=1}^n q_i^2 x_i^2}\, d\sigma.$$

where $\mathbf{q}$ is the vector of inverses of the major semi-axes of E. We know that for any $p > 0$,

$$c_{n,p}||q||_p \leq ||q||_{\mathbb{R}} \leq C_{n,p}||q||_p,$$

for some dimensional constants $c_{n,p}, C_{n,p}$. In this section we will give good (estimates on the constants $c_{n,2}$ and $C_{n,2}$.

To estimate $C_{n,2}$ we will first show the following:

Lemma 13. *Let $F(x)$ be a probability distribution, and let*

$$M_a(F) \stackrel{def}{=} \mathbb{E}_F(|x|^a)$$

denote the absolute moments of F (we will abuse notation in the sequel by referring to the absolute moments of a random variable as well as those of its distribution function). Further, let $0 \leq \beta \leq \alpha$. Then

$$M_\beta(F) \leq 1 + M_\alpha(F).$$

Proof.

$$M_\beta(F) = \int_{-\infty}^{\infty} |x|^\beta \, dF = \int_{-1}^{1} |x|^\beta \, dF + \int_{-\infty}^{-1} |x|^\beta \, dF + \int_{1}^{\infty} |x^\beta \, dF.$$

We note that $\int_{-1}^{1} |x|^\beta \, dF \leq \int_{-1}^{1} dF \leq 1$, while $\int_{-\infty}^{-1} |x|^\beta \, dF \leq \int_{-\infty}^{0} |x|^\alpha \, dF$, and similarly with the integral from 1 to ∞. The assertion of the Lemma is then immediate. □

Theorem 14. *Let $X_1, \ldots, X_n$ be* positive *i. i. d. random variables, with finite mean μ. Let $a_1, \ldots, a_n$ be non-negative coefficients. Then*

$$S = \mathbb{E}(\sqrt{\sum_{i=1}^n a_i X_i}) \leq (1+\mu)\sqrt{\sum_{i=1}^n a_i}.$$

Proof. Let

$$Y = \sum_{i=1}^n a_i X_i;$$

it follows that $S = M_{1/2}(Y)$. Let

$$Z = \frac{Y}{\sum_{i=1}^n a_i}.$$

We see that $M_1(Z) = \mu$. By Lemma 13, it follows that

$$M_{1/2}(Z) \leq \mu + 1,$$

and thus

$$S = M_{1/2}(Y) \leq (1+\mu)\sqrt{\sum_{i=1}^n a_i}.$$

□

Corollary 15.

$$C_{n,2} \leq \frac{3\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof. Replace a_i by q_i^2 and make the variables X_i squares of centered Gaussians with variance 1/2 in the statement of Theorem 14, and apply the results of Section 2 to get the quotient of Γ values. □

To estimate $c_{n,2}$ we will first note:

Lemma 16. *Let $\lambda_1, \dots, \lambda_n \geq 0$, and let $\lambda_1 + \cdots + \lambda_n = 1$. Then*

$$\sqrt{\sum_{i=1}^n \lambda_i x_i^2} \geq \sum_{i=1}^n \lambda_i |x_i|.$$

Proof. Concavity of square root. □

Corollary 17.

$$\fint_{\mathbb{S}^{n-1}} \sqrt{\sum_{i=1}^n q_i^2 u_i^2}\, d\sigma \geq \fint_{\mathbb{S}^{n-1}} |x_1|\, d\sigma \sqrt{\sum_{i=1}^n q_i^2}.$$

Proof. Write

$$\sum_{i=1}^n q_i^2 u_i^2 = \sum_{i=1}^n q_i^2 \sum_{i=1}^n \frac{q_i^2}{\sum_{j=1}^n q_j^2} x_i^2,$$

then use the symmetry to note that

$$\fint_{\mathbb{S}^{n-1}} x_i\, d\sigma == \fint_{\mathbb{S}^{n-1}} x_j\, d\sigma$$

for any i, j. □

Corollary 18.

$$c_{n,2} \geq \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof. Immediate from Corollary 17 □

Combining Corollary 15 and Corollary 18 we get

$$\frac{3\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)} \geq \frac{\|q\|_{\mathbb{R}}}{\|q\|} \geq \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}. \tag{15}$$

Note that the ratio between the right and the left hand sides of the inequality (15) stays bounded, so this inequality is sharp to within a constant factor. It is fairly clear that the constants are not sharp, however.

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